

ON A_p - A_∞ TYPE ESTIMATES FOR SQUARE FUNCTIONS

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ABSTRACT. We prove strong-type A_p - A_∞ estimate for square functions, improving on the A_p bound due to Lerner. Entropy bounds, in the recent innovation of Treil-Volberg, are then proved. The techniques of proof include parallel stopping cubes, pigeon-hole arguments, and the approach to entropy bounds of Lacey–Spencer.

1. INTRODUCTION

What are the weakest ‘ A_p like’ conditions that are sufficient for two weight inequalities for square functions? Replacing square functions by Calderón-Zygmund operators, this question has received wide ranging attention since the birth of the weighted theory. The finest results known are the A_p - A_∞ bounds [5]; the mixed A_p - A_r inequalities of Lerner [10], and the entropy bounds of Treil-Volberg [13], and the weak-type bounds of [1]. We refer the reader to the introductions of these papers for a guide to the long history of this question. The analog of these results for strong-type bounds for square functions are the focus of this paper. (The reference [1] also includes weak-type estimates for square functions.)

We begin with the definition of the intrinsic square functions introduced by Wilson [15]. For $0 < \alpha \leq 1$, let \mathcal{C}_α be the family of functions supported in $\{x : |x| \leq 1\}$, satisfying $\int \varphi = 0$, and such that for all x and x' , $|\varphi(x) - \varphi(x')| \leq |x - x'|^\alpha$. If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $(y, t) \in \mathbb{R}^{n+1}_+$, we define

$$A_\alpha(f)(y, t) = \sup_{\varphi \in \mathcal{C}_\alpha} |f * \varphi_t(y)|.$$

Then the intrinsic square function is defined by

$$G_{\beta, \alpha}(f)(x) = \left(\int_{\Gamma_\beta(x)} (A_\alpha(f)(y, t))^2 \frac{dy dt}{t^{n+1}} \right)^{1/2},$$

where $\Gamma_\beta(x) = \{(y, t) : |y - x| < \beta t\}$. If $\beta = 1$, set $G_{1, \alpha}(f) = G_\alpha(f)$. Wilson showed that $G_{\beta, \alpha}(f) \sim G_\alpha(f)$ and it dominates the continuous type square functions including the Lusin area function and Littlewood-Paley g function. Therefore, we only focus on $G_\alpha(f)$.

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In [9], Lerner gave the following estimate

$$(1.1) \quad \|G_\alpha\|_{L^p(w)} \leq c(G_\alpha, n, p)[w]_{A_p}^{\max\{\frac{1}{2}, \frac{1}{p-1}\}}.$$

This estimate is sharp in the exponent of $[w]_{A_p}$ and hence is the square function analog of the A_2 bound for Calderón-Zygmund operators, proved by Hytönen [3].

Lerner [10], has established mixed A_p - A_r estimates when $p \geq 3$. These estimates only involve a single supremum to define, and are restricted to the one weight setting. The interested reader should refer to [10].

Our focus is on the two weight A_p - A_∞ type estimates for square functions. Given a pair of weights w and σ , define $\langle w \rangle_Q := \frac{1}{|Q|} \int_Q w(x) dx$,

$$[w, \sigma]_{A_p} := \sup_Q \langle w \rangle_Q \langle \sigma \rangle_Q^{p-1},$$

$$\text{and} \quad [w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w \mathbf{1}_Q) dx.$$

Our first result is the following

Theorem 1.2. *Given $1 < p < \infty$. Let w and σ be a pair of weights such that $[w, \sigma]_{A_p} < \infty$ and $w, \sigma \in A_\infty$. Then*

$$(1.3) \quad \|G_\alpha(\cdot\sigma)\|_{L^p(\sigma) \rightarrow L^p(w)} \lesssim \begin{cases} [w, \sigma]_{A_p}^{\frac{1}{p}} [\sigma]_{A_\infty}^{\frac{1}{p}}, & 1 < p \leq 2, \\ [w, \sigma]_{A_p}^{\frac{1}{p}} ([w]_{A_\infty}^{\frac{1}{2} - \frac{1}{p}} + [\sigma]_{A_\infty}^{\frac{1}{p}}), & p > 2. \end{cases}$$

where the constant C is independent of the weights w and σ .

Specializing this to the one weight case, we have $[\sigma]_{A_\infty} \lesssim [\sigma]_{A_{p'}} = [w]_{A_p}^{\frac{1}{p-1}}$, and so Lerner's bound (1.1) follows from the Theorem, as can be checked by elementary considerations. This is interesting, since the inequalities (1.3) have $p = 2$ as a critical index, while Lerner's bound has $p = 3$ as the critical index. We find that this reflected in the proof of the result above, with one term splitting neatly at $p = 2$, and another splitting at $p = 2$ and at $p = 3$.

Concerning the proof, we will use the common reduction to a positive sparse square function. In the two weight setting, we have a characterization of the required inequality in terms of (quadratic) testing assumptions [12, 2, 14]. These conditions are however difficult to work with. Instead, we use the parallel stopping cubes introduced Lacey, Sawyer, Shen and Uriarte-Tuero [7], as elaborated in the last section of [4].

The second topic is to prove entropy bounds for the square function. Here, we are using the recent innovative approach of Treil-Volberg [13], which is an improvement over the Orlicz norm approach to 'bumping', started by C. Pérez [11]. Again, the reader should consult [13] for a history of this point of view, and an explanation of why the entropy method is stronger than that of Orlicz norm approach.

There is a very close connection between the entropy bounds and A_∞ bounds, a feature exploited by Lacey-Spencer [8]. The entropy conditions are given in terms of the ‘local A_∞ ’ constant, which is allowed to take arbitrarily large values, at the cost of a logarithmic penalty.

$$\rho_\sigma(Q) = \frac{\int_Q M(\sigma \mathbf{1}_Q) dx}{\sigma(Q)} \quad \text{and} \quad \rho_{\sigma,\epsilon}(Q) = \rho_\sigma(Q) \epsilon(\rho_\sigma(Q)),$$

where ϵ is a monotonic increasing function on $(1, \infty)$. The ϵ gives the penalty on a locally large A_∞ constant. The entropy conditions then come in two forms, one in which both weights are ‘bumped’ in a multiplicative manner,

$$[w, \sigma]_{p,\epsilon} := \begin{cases} \sup_Q \langle w \rangle_Q^{\frac{1}{p}} \langle \sigma \rangle_Q^{\frac{1}{p'}} \rho_{\sigma,\epsilon}(Q)^{\frac{1}{p}}, & 1 < p \leq 2, \\ \sup_Q \langle w \rangle_Q^{\frac{1}{p}} \langle \sigma \rangle_Q^{\frac{1}{p'}} \rho_{w,\epsilon}(Q)^{\frac{1}{2} - \frac{1}{p}} \rho_{\sigma,\epsilon}(Q)^{\frac{1}{p}}, & p > 2. \end{cases}$$

and the other in an additive or ‘separated’ fashion,

$$[w, \sigma]_{p,\epsilon,\eta} := \begin{cases} \sup_Q \langle w \rangle_Q^{\frac{1}{p}} \langle \sigma \rangle_Q^{\frac{1}{p'}} \rho_{\sigma,\epsilon}(Q)^{\frac{1}{p}}, & 1 < p \leq 2, \\ \sup_Q \langle w \rangle_Q^{\frac{1}{p}} \langle \sigma \rangle_Q^{\frac{1}{p'}} (\rho_{w,\eta}(Q)^{\frac{1}{2} - \frac{1}{p}} + \rho_{\sigma,\epsilon}(Q)^{\frac{1}{p}}), & p > 2, \end{cases}$$

where η is another monotonic increasing function on $(1, \infty)$. Now we are ready to state our entropy bounds for square functions. These inequalities are the analogs of the main results in [8], and the proof is along the lines of that paper.

Theorem 1.4. *Let (w, σ) be a pair of weights and ϵ, η be two monotonic increasing functions on $(1, \infty)$. If ϵ satisfy*

$$\begin{cases} \int_1^\infty \frac{1}{t\epsilon(t)^{1/p}} dt < \infty, & 1 < p \leq 2 \\ \int_1^\infty \frac{1}{t\epsilon(t)} dt < \infty, & p > 2, \end{cases}$$

then there holds

$$\|G_\alpha(\cdot\sigma)\|_{L^p(\sigma) \rightarrow L^p(w)} \lesssim [w, \sigma]_{p,\epsilon}.$$

For any $1 < p < \infty$ and ϵ, η satisfy

$$\begin{cases} \int_1^\infty \frac{1}{t\epsilon(t)^{1/p}} dt < \infty, & 1 < p \leq 2 \\ \int_1^\infty \frac{1}{t\epsilon(t)^{1/p}} dt + \int_1^\infty \frac{1}{t\eta(t)^{\frac{1}{2} - \frac{1}{p}}} dt < \infty, & p > 2. \end{cases}$$

then there holds

$$\|G_\alpha(\cdot\sigma)\|_{L^p(\sigma) \rightarrow L^p(w)} \lesssim [w, \sigma]_{p,\epsilon,\eta}.$$

2. PROOF OF THEOREM 1.2

The first step is, as is fundamental in this subject, the reduction to positive sparse operators. A collection of dyadic cubes \mathcal{S} is said to be *sparse* if for all $Q \in \mathcal{S}$,

$$\left| \bigcup_{\substack{Q', Q \in \mathcal{S} \\ Q' \subsetneq Q}} Q' \right| \leq \frac{1}{2} |Q|.$$

Then the positive sparse operator related to \mathcal{S} is defined by

$$A_{\mathcal{S}}(f) = \left(\sum_{Q \in \mathcal{S}} \langle f \rangle_Q^2 \mathbf{1}_Q \right)^{1/2}.$$

It is well known that there are many dyadic grids, and moreover, there are at most 3^n choices of dyadic grids in \mathbb{R}^n so that *any* cube in \mathbb{R}^n is well-approximated by a choice of cube from one of the specified dyadic grids.

The following lemma is a variant of the argument in [6].

Lemma 2.1. *If f is bounded and compactly supported, there are at most 3^n sparse collections of dyadic cubes \mathcal{S}_j , $1 \leq j \leq 3^n$, so that the pointwise inequality below holds.*

$$G_{\alpha} f \lesssim \sum_{j=1}^{3^n} A_{\mathcal{S}_j} f.$$

Clearly, we need only consider the weighted bounds for a single sparse operator. There is then nothing special about the ℓ^2 sum used to define the operator, hence we define ℓ^r variants as follows.

$$(2.2) \quad \left(A_{\mathcal{S}}^r f \right)^r = \sum_{Q \in \mathcal{S}} \langle f \rangle_Q^r \mathbf{1}_Q.$$

We have the following more general estimate.

Theorem 2.3. *Let $A_{\mathcal{S}}^r$ be defined as in (2.2) and $p > r$, then*

$$\|A_{\mathcal{S}}^r(\cdot\sigma)\|_{L^p(\sigma) \rightarrow L^p(w)} \leq C[w, \sigma]_{A_p}^{\frac{1}{p}} \left([w]_{A_{\infty}}^{\frac{1}{r} - \frac{1}{p}} + [\sigma]_{A_{\infty}}^{\frac{1}{p}} \right)$$

where the constant C is independent of w and σ .

It is a useful remark that the estimate above can be made slightly more precise, in that the supremums defining the two-weight A_p and A_{∞} constants need only be taken over the collection of cubes \mathcal{S} . To be precise, $[w, \sigma]_{A_p}$ above can be replaced by

$$(2.4) \quad [w, \sigma]_{A_p, \mathcal{S}} := \sup_{Q \in \mathcal{S}} \langle \sigma \rangle_Q^{p-1} \langle w \rangle_Q,$$

and similarly for $[w]_{A_{\infty}}$ can be replaced by $[w]_{A_{\infty}, \mathcal{S}}$, which has a similar definition.

Proof. Use duality to eliminate the r th root. Since $p > r$, it suffices to prove

$$(2.5) \quad \langle (A_{\mathcal{S}}^r f \sigma)^r, gw \rangle \lesssim N,$$

under these conditions.

- (1) $N^{1/r}$ satisfies the bounds of the Theorem.
- (2) The functions f and g are normalized so that $\|f\|_{L^p(\sigma)} = 1$ and $\|g\|_{L^q(w)} = 1$, where $q = (p/r)' = \frac{p}{p-r}$.
- (3) The sparse collection \mathcal{S} satisfies

$$(2.6) \quad 2^{a-1} < \langle \sigma \rangle_Q^{p-1} \langle w \rangle_Q \leq 2^a,$$

for some integer a . (And then one can sum over a .)

- (4) All cubes $Q \in \mathcal{S}$ are contained in a root cube Q^0 .

The parallel corona is used to decompose the inner product in (2.5). Now we can define the principal cubes \mathcal{F} for (f, σ) and \mathcal{G} for (g, w) . Namely,

$$\begin{aligned} \mathcal{F} &:= \bigcup_{k=0}^{\infty} \mathcal{F}_k, \quad \mathcal{F}_0 := \{\text{maximal cubes in } \mathcal{S}\} \\ \mathcal{F}_{k+1} &:= \bigcup_{F \in \mathcal{F}_k} \text{ch}_{\mathcal{F}}(F), \quad \text{ch}_{\mathcal{F}}(F) := \{Q \subsetneq F \text{ maximal s.t. } \langle f \rangle_Q^\sigma > 2\langle f \rangle_F^\sigma\}, \end{aligned}$$

and analogously for \mathcal{G} . We also denote by $\pi_{\mathcal{F}}(Q)$ the minimal cube in \mathcal{F} which contains Q , and $\pi(Q) = (F, G)$ if $\pi_{\mathcal{F}}(Q) = F$ and $\pi_{\mathcal{G}}(Q) = G$. With this definition, it is easy to check that for any $1 < p < \infty$,

$$(2.7) \quad \sum_{F \in \mathcal{F}} (\langle f \rangle_F^\sigma)^p \sigma(F) \lesssim \|f\|_{L^p(\sigma)}^p$$

and a similar inequality holds for g .

In terms of the principal cubes, $\langle (A_{\mathcal{S}}^r f \sigma)^r, gw \rangle$ is less than 2^{r+1} times the sum $I + II$, where

$$\begin{aligned} I &:= \sum_{F \in \mathcal{F}} (\langle f \rangle_F^\sigma)^r \underbrace{\sum_{\substack{G \in \mathcal{G} \\ \pi_{\mathcal{F}}(G)=F}} \langle g \rangle_G^w \sum_{\substack{Q \in \mathcal{S} \\ \pi(Q)=(F,G)}} \langle \sigma \rangle_Q^r w(Q)}_{:=I(F)}, \\ II &:= \sum_{G \in \mathcal{G}} \langle g \rangle_G^w \underbrace{\sum_{\substack{F \in \mathcal{F} \\ \pi_{\mathcal{G}}(F)=G}} (\langle f \rangle_F^\sigma)^r \sum_{\substack{Q \in \mathcal{S} \\ \pi(Q)=(F,G)}} \langle \sigma \rangle_Q^r w(Q)}_{:=II(G)}. \end{aligned}$$

In view of (2.7), the bound for $I(F)$ we need is of the form below.

$$(2.8) \quad I(F) \lesssim N_I \sigma(F)^{\frac{r}{p}} \left[\sum_{\substack{G \in \mathcal{G} \\ \pi_{\mathcal{F}}(G)=F}} (\langle g \rangle_G^w)^q w(G) \right]^{\frac{1}{q}}, \quad F \in \mathcal{F}.$$

Recall that $\frac{1}{q} = \frac{p-r}{p}$, so that $\frac{1}{q} + \frac{r}{p} = 1$. Indeed, with this bound, a straight forward application of Hölder's inequality, with (2.7) completes a proof of (2.5). We then conclude that $I \lesssim N_I$. For $II(G)$, the bound is of the form below.

$$(2.9) \quad II(G) \lesssim N_{II} w(G)^{\frac{1}{q}} \left[\sum_{\substack{F \in \mathcal{F} \\ \pi_{\mathcal{G}}(F)=G}} (\langle f \rangle_F^\sigma)^r \sigma(F) \right]^{\frac{r}{p}}, \quad G \in \mathcal{G}.$$

Let us now bound N_I , for all $r < p < \infty$. Observe that

$$\begin{aligned} I(F) &\lesssim \sum_{\substack{G \in \mathcal{G} \\ \pi_{\mathcal{F}}(G)=F}} \sum_{\substack{Q \in \mathcal{S} \\ \pi(Q)=(F,G)}} \int_Q \left(\sup_{\substack{G' \in \mathcal{G} \\ \pi_{\mathcal{F}}(G')=F}} \langle g \rangle_{G'}^w \mathbf{1}_{G'} \right) \langle \sigma \rangle_Q^r \mathbf{1}_Q dw \\ &\leq \int_F \left(\sup_{\substack{G' \in \mathcal{G} \\ \pi_{\mathcal{F}}(G')=F}} \langle g \rangle_{G'}^w \mathbf{1}_{G'} \right) A_{\mathcal{S}(F)}^r (\sigma \mathbf{1}_F)^r dw \\ &\leq \left\| \sup_{G' \in \mathcal{G}(F)} \langle g \rangle_{G'}^w \mathbf{1}_{G'} \right\|_{L^q(w)} \cdot \|A_{\mathcal{S}(F)}^r (\sigma \mathbf{1}_F)^r\|_{L^{p/r}(w)}. \end{aligned}$$

Now, the first term on the right, by construction of the principal cubes is no more than

$$\left[\sum_{\substack{G \in \mathcal{G} \\ \pi_{\mathcal{F}}(G)=F}} (\langle g \rangle_G^w)^q w(G) \right]^{\frac{1}{q}}$$

as required in (2.8). The second term, the notation is $\mathcal{S}(F) = \{Q \in \mathcal{S} : \pi_{\mathcal{F}}(Q) = F\}$. Below, we dominate ℓ^r -norms by ℓ^1 , and appeal to [5, Prop. 5.3] to see that

$$\|A_{\mathcal{S}(F)}^r (\sigma \mathbf{1}_F)^r\|_{L^{p/r}(w)} \leq \|A_{\mathcal{S}(F)}^1 (\sigma \mathbf{1}_F)\|_{L^p(w)}^r \lesssim [[w, \sigma]_{A_p, \mathcal{S}(F)} [\sigma]_{A_\infty, \mathcal{S}(F)} \sigma(F)]^{\frac{r}{p}}.$$

Our conclusion is that

$$(2.10) \quad N_I(F) \lesssim 2^{a \frac{r}{p}} [\sigma]_{A_\infty}^{\frac{r}{p}}.$$

Now we turn to the analysis of $II(G)$. This is the more delicate case, that breaks into the two subcases of $r < p \leq r+1$, and $r+1 \leq p$, though the resulting inequality is the same in both cases. We treat the case of $r < p < r+1$ first. The first step is to again appeal to (2.6) to write

$$\begin{aligned} \langle \sigma \rangle_Q^r w(Q) &\simeq \langle \sigma \rangle_Q^r \langle w \rangle_Q \cdot |Q| \\ &\simeq 2^a \langle \sigma \rangle_Q^{r+1-p} |Q|. \end{aligned}$$

Indeed, in the last line, we can replace $|Q|$ by $|E(Q)|$, the exceptional set associated to Q . The sets $E(Q)$ are disjoint in Q , whence

$$\sum_{\substack{Q \in \mathcal{S} \\ \pi(Q)=(F,G)}} \langle \sigma \rangle_Q^r w(Q) \lesssim 2^a \int_F M(\sigma \mathbf{1}_F)^{r+1-p} dx.$$

Now, recall that on probability spaces that L^t norms increase in t . This has an extension to Lorentz spaces, from which we conclude that

$$\begin{aligned} \left[\frac{1}{|F|} \int_F M(\sigma \mathbf{1}_F)^{r+1-p} dx \right]^{\frac{1}{r+1-p}} &\leq \| \mathbf{1}_F M(\sigma \mathbf{1}_F) \|_{L^{1,\infty}(F, \frac{dx}{|F|})} \\ &\lesssim \langle \sigma \rangle_F. \end{aligned}$$

This just depends upon the maximal function bound. Simplifying, we have the bound

$$\begin{aligned} \sum_{\substack{Q \in \mathcal{S} \\ \pi(Q)=(F,G)}} \langle \sigma \rangle_Q^r w(Q) &\lesssim 2^a \sigma(F)^{r+1-p} |F|^{p-r} \\ &\lesssim 2^a 2^{a \frac{r-p}{p}} \sigma(F)^{r+1-p} [\sigma(F)^{p-1} w(F)]^{\frac{1}{q}}. \end{aligned}$$

Here, we must note that the power on $\sigma(F)$ is $(\frac{1}{q} = \frac{p-r}{p})$

$$r+1-p+(p-1)\frac{p-r}{p} = \frac{r}{p}.$$

Indeed, this is easy to see by multiplying both sides above by p .

It follows that

$$\begin{aligned} II(G) &\lesssim 2^{a \frac{r}{p}} \sum_{\substack{F \in \mathcal{F} \\ \pi_G(F)=G}} (\langle f \rangle_F^\sigma)^r \sigma(F)^{\frac{r}{p}} w(F)^{\frac{1}{q}} \\ &\lesssim 2^{a \frac{r}{p}} [w]_{A_\infty}^{\frac{p-r}{p}} w(G)^{\frac{1}{q}} \left[\sum_{\substack{F \in \mathcal{F} \\ \pi_G(F)=G}} (\langle f \rangle_F^\sigma)^p \sigma(F) \right]^{\frac{r}{p}}, \quad r < p < r+1. \end{aligned}$$

This just depends upon an application of Hölder's inequality, and an appeal to the A_∞ constant of w to bound the sum over F of $w(F)$. We conclude the bound below, which is just as the Theorem claims.

$$(2.11) \quad N_{II} \lesssim 2^{a \frac{r}{p}} [w]_{A_\infty}^{\frac{p-r}{p}}, \quad r < p < r+1.$$

The last case is to estimate N_{II} in the case of $r+1 \leq p < \infty$. We begin by eliminating the $\sigma(Q)^r$ in the term on the right below: By (2.6)

$$\langle \sigma \rangle_Q^r w(Q) \simeq 2^{\frac{ar}{p-1}} w(Q)^{1-\frac{r}{p-1}} |Q|^{\frac{r}{p-1}}.$$

The exponents above are in Hölder's duality, thus

$$\begin{aligned} \sum_{Q: \pi(Q)=(F,G)} \langle \sigma \rangle_Q^r w(Q) &\lesssim 2^{\frac{ar}{p-1}} |F|^{\frac{r}{p-1}} \left[\sum_{Q: \pi(Q)=(F,G)} w(Q) \right]^{1-\frac{r}{p-1}} \\ &\lesssim 2^{a \frac{r}{p}} \sigma(F)^{\frac{r}{p}} \left[\sum_{Q: \pi(Q)=(F,G)} w(Q) \right]^{1-\frac{r}{p-1}} w(F)^{\frac{r}{p(p-1)}}. \end{aligned}$$

In the second line, appeal to (2.6) again, converting $|F|$ in the first line into a geometric mean of $\sigma(F)$ and $w(F)$.

Therefore, from the definition of $II(G)$, we have

$$\begin{aligned} II(G) &\lesssim 2^{a\frac{r}{p}} \sum_{F: \pi_G(F)=G} (\langle f \rangle_F^\sigma)^r \sigma(F)^{\frac{r}{p}} \left[\sum_{Q: \pi(Q)=(F,G)} w(Q) \right]^{1-\frac{r}{p-1}} w(F)^{\frac{r}{p(p-1)}} \\ &\lesssim 2^{a\frac{r}{p}} [w]_{A_\infty}^{\frac{1}{q}} \left[\sum_{F: \pi_G(F)=G} (\langle f \rangle_F^\sigma)^p \sigma(F) \right]^{\frac{r}{p}} w(G)^{\frac{1}{q}} \end{aligned}$$

after an application of the trilinear form of Hölder's inequality, and the use of the A_∞ property of w . We conclude that

$$(2.12) \quad N_{II} \lesssim 2^{a\frac{r}{p}} [w]_{A_\infty}^{\frac{p-r}{p}}, \quad r+1 \leq p < \infty.$$

The proof follows by combining the estimates (2.8), (2.10), (2.9), (2.11) and (2.12). \square

The case of $1 < p \leq r$ is quite simple, we have the following estimate

Theorem 2.13. *Let $1 < p \leq r$. There holds*

$$\|A_S^r(\cdot\sigma)\|_{L^p(\sigma) \rightarrow L^p(w)} \leq C[w, \sigma]_{A_p}^{\frac{1}{p}} [\sigma]_{A_\infty}^{\frac{1}{p}},$$

where the constant C is independent of the weights w and σ .

Proof. We only need principle cubes for the function f , and we use the same definition and notation from the previous proof. Since ℓ^p norms are larger than ℓ^r norms, we have

$$\begin{aligned} \|A_S^r(f\sigma)\|_{L^p(w)}^p &= \left\| \left(\sum_{F \in \mathcal{F}} \sum_{\substack{Q \in \mathcal{S} \\ \pi(Q)=F}} (\langle f \rangle_Q^\sigma)^r \langle \sigma \rangle_Q^r \mathbf{1}_Q \right)^{\frac{1}{r}} \right\|_{L^p(w)}^p \\ &\lesssim \sum_{F \in \mathcal{F}} (\langle f \rangle_F^\sigma)^p \sum_{\substack{Q \in \mathcal{S} \\ \pi(Q)=F}} \langle \sigma \rangle_Q^p w(Q) \\ &\lesssim [w, \sigma]_{A_p} \sum_{F \in \mathcal{F}} (\langle f \rangle_F^\sigma)^p \sum_{\substack{Q \in \mathcal{S} \\ \pi(Q)=F}} \sigma(Q) \\ &\lesssim [w, \sigma]_{A_p} [\sigma]_{A_\infty} \|f\|_{L^p(\sigma)}^p \end{aligned}$$

where we have used the A_p and A_∞ properties in a straight forward way. \square

Now with Theorems 2.3 and 2.13, Theorem 1.2 follows immediately, by setting $r = 2$.

3. PROOF OF THEOREM 1.4

In this section, we shall give a proof for Theorem 1.4. In the proof, we will have recourse to this Carleson embedding inequality. Proved in [13, Theorem 4.2], it has a very short proof given in [8, §4].

Lemma 3.1. *Let ϵ be a monotonic increasing function on $(1, \infty)$ such that $\int_1^\infty \frac{dt}{\epsilon(t)t} < \infty$. Then for all $1 < p < \infty$, we have*

$$\sum_{Q \in \mathcal{S}} (\langle f \rangle_Q^\sigma)^p \frac{\sigma(Q)}{\rho_{\sigma, \epsilon}(Q)} \lesssim \|f\|_{L^p(\sigma)}^p.$$

Again, we reduce the problem to consider the sparse operators and we shall show the following more general estimate.

Theorem 3.2. *Let A_S^r be defined as in (2.2) and ϵ, η be two monotonic increasing functions on $(1, \infty)$. Then if $p > r$ and $\int_1^\infty \frac{dt}{\epsilon(t)t} < \infty$, we have the entropy bound*

$$(3.3) \quad \|A_S^r(\cdot\sigma)\|_{L^p(\sigma) \rightarrow L^p(w)} \lesssim \sup_Q \langle w \rangle_Q^{\frac{1}{p}} \langle \sigma \rangle_Q^{\frac{1}{p'}} \rho_{w, \epsilon}(Q)^{\frac{1}{r} - \frac{1}{p}} \rho_{\sigma, \epsilon}(Q)^{\frac{1}{p}}.$$

If $p > r$ and ϵ, η satisfy

$$\int_1^\infty \frac{1}{t\epsilon(t)^{1/p}} dt + \int_1^\infty \frac{1}{t\eta(t)^{\frac{1}{r} - \frac{1}{p}}} dt < \infty,$$

we also have the separated entropy bound

$$(3.4) \quad \|A_S^r(\cdot\sigma)\|_{L^p(\sigma) \rightarrow L^p(w)} \lesssim \sup_Q \langle w \rangle_Q^{\frac{1}{p}} \langle \sigma \rangle_Q^{\frac{1}{p'}} (\rho_{w, \eta}(Q)^{\frac{1}{r} - \frac{1}{p}} + \rho_{\sigma, \epsilon}(Q)^{\frac{1}{p}}).$$

Proof. First, we prove (3.3). Denote

$$[w, \sigma]_{p, r, \epsilon} = \sup_Q \langle w \rangle_Q^{\frac{1}{p}} \langle \sigma \rangle_Q^{\frac{1}{p'}} \rho_{w, \epsilon}(Q)^{\frac{1}{r} - \frac{1}{p}} \rho_{\sigma, \epsilon}(Q)^{\frac{1}{p}}.$$

Follow the method used in [8], set

$$\mathcal{Q}_a = \{Q \in \mathcal{S} : 2^a < \langle w \rangle_Q^{\frac{1}{p}} \langle \sigma \rangle_Q^{\frac{1}{p'}} \rho_{w, \epsilon}(Q)^{\frac{1}{r} - \frac{1}{p}} \rho_{\sigma, \epsilon}(Q)^{\frac{1}{p}} \leq 2^{a+1}\},$$

here $2^a \leq [w, \sigma]_{p, r, \epsilon}$. Recall that $q = (p/r)'$, by duality, we have

$$\|A_S^r(f\sigma)\|_{L^p(w)}^r = \sup_{\|g\|_{L^{q'}(w)}=1} \sum_{Q \in \mathcal{S}} \langle f\sigma \rangle_Q^r \int_Q g dw.$$

Now fix $g \in L^{q'}(w)$ with $\|g\|_{L^{q'}(w)} = 1$. We have for $Q \in \mathcal{Q}_a$,

$$\begin{aligned} \langle f\sigma \rangle_Q^r \int_Q g dw &= (\langle f \rangle_Q^\sigma)^r \langle g \rangle_Q^w \langle \sigma \rangle_Q^r w(Q) \\ &\lesssim 2^{ar} (\langle f \rangle_Q^\sigma)^r \frac{\sigma(Q)^{\frac{r}{p}}}{\rho_{\sigma, \epsilon}(Q)^{\frac{r}{p}}} \cdot \langle g \rangle_Q^w \frac{w(Q)^{\frac{1}{q}}}{\rho_{w, \epsilon}(Q)^{\frac{1}{q}}}. \end{aligned}$$

The indices are set up for an application of Hölder's inequality. The sum over Q of the terms above is

$$\begin{aligned}
& \sum_{a \leq \log_2 \lceil w, \sigma \rceil_{p,r,\epsilon}} \sum_{Q \in \mathcal{Q}_a} \langle f \sigma \rangle_Q^r \int_Q g dw \\
& \lesssim \sum_{a \leq \log_2 \lceil w, \sigma \rceil_{p,r,\epsilon}} 2^{ar} \left[\sum_{Q \in \mathcal{Q}_a} (\langle f \rangle_Q^\sigma)^p \frac{\sigma(Q)}{\rho_{\sigma,\epsilon}(Q)} \right]^{r/p} \left[\sum_{Q \in \mathcal{Q}_a} (\langle g \rangle_Q^w)^{q'} \frac{w(Q)}{\rho_{w,\epsilon}(Q)} \right]^{\frac{1}{q}} \\
& \lesssim \lceil w, \sigma \rceil_{p,r,\epsilon}^r \|f\|_{L^p(\sigma)}^r.
\end{aligned}$$

Lemma 3.1 is used in the last step, to control the sums involving both f and g .

Next we consider (3.4). We decompose the collection \mathcal{S} into subsets $\mathcal{S}_{a,b}$ and $\mathcal{S}'_{a,b}$, for integers a, b . The collection $\mathcal{S}_{a,b}$ consists of those $Q \in \mathcal{S}$ which meet three conditions,

$$\begin{aligned}
2^a &< \langle w \rangle_Q^{\frac{1}{p}} \langle \sigma \rangle_Q^{\frac{1}{p'}} \rho_{\sigma,\epsilon}(Q)^{\frac{1}{p}} \leq 2^{a+1} \\
\rho_\sigma(Q)^{\frac{1}{p}} &\geq \rho_w(Q)^{\frac{1}{r} - \frac{1}{p}}, \\
2^b &< \rho_\sigma(Q) \leq 2^{b+1}.
\end{aligned}$$

We also denote $\mathcal{S}'_{a,b}$ the sub-collection of \mathcal{S} such that

$$\begin{aligned}
2^a &< \langle w \rangle_Q^{\frac{1}{p}} \langle \sigma \rangle_Q^{\frac{1}{p'}} \rho_{w,\eta}(Q)^{\frac{1}{r} - \frac{1}{p}} \leq 2^{a+1}, \\
\rho_\sigma(Q)^{\frac{1}{p}} &< \rho_w(Q)^{\frac{1}{r} - \frac{1}{p}}, \\
2^b &< \rho_w(Q) \leq 2^{b+1}.
\end{aligned}$$

Every $Q \in \mathcal{S}$ is in either $\mathcal{S}_{a,b}$ or $\mathcal{S}'_{a,b}$ for some choice of a, b , and are empty if either $b \leq -1$, or $2^a > 2 \lceil w, \sigma \rceil_{p,r,\epsilon,\eta}$.

The initial estimate is then as below, where it is important to note that we are using the quantification of Theorem 2.3, as described in (2.4).

$$\begin{aligned}
\|A_{\mathcal{S}}^r(f\sigma)\|_{L^p(w)} &\leq \sum_{a,b} \|A_{\mathcal{S}_{a,b}}^r(f\sigma)\|_{L^p(w)} + \|A_{\mathcal{S}'_{a,b}}^r(f\sigma)\|_{L^p(w)} \\
&\leq \sum_{a,b} [w, \sigma]_{A_p, \mathcal{S}_{a,b}}^{\frac{1}{p}} ([w]_{A_\infty, \mathcal{S}_{a,b}}^{\frac{1}{r} - \frac{1}{p}} + [\sigma]_{A_\infty, \mathcal{S}_{a,b}}^{\frac{1}{p}}) \\
&\quad + \sum_{a,b} [w, \sigma]_{A_p, \mathcal{S}'_{a,b}}^{\frac{1}{p}} ([w]_{A_\infty, \mathcal{S}'_{a,b}}^{\frac{1}{r} - \frac{1}{p}} + [\sigma]_{A_\infty, \mathcal{S}'_{a,b}}^{\frac{1}{p}}) \\
&:= I + II.
\end{aligned}$$

First, we estimate I . By definition of $\mathcal{S}_{a,b}$, there holds

$$[w]_{A_\infty, \mathcal{S}_{a,b}}^{\frac{1}{r} - \frac{1}{p}} \leq [\sigma]_{A_\infty, \mathcal{S}_{a,b}}^{\frac{1}{p}},$$

so that

$$\begin{aligned} I &\lesssim \sum_{a,b} [w, \sigma]_{A_p, \mathcal{S}_{a,b}}^{\frac{1}{p}} [\sigma]_{A_\infty, \mathcal{S}_{a,b}}^{\frac{1}{p}} \\ &\lesssim \sum_{a,b} \frac{2^a}{2^{b/p} \epsilon (2^b)^{1/p}} 2^{b/p} \\ &\lesssim N \int_1^\infty \frac{1}{t \epsilon(t)^{1/p}} dt, \end{aligned}$$

where N denotes the right side of (3.4). The dual term follows an analogous line of reasoning, leading to the estimate below, which completes the proof.

$$II \lesssim N \int_1^\infty \frac{1}{t \eta(t)^{\frac{1}{r} - \frac{1}{p}}} dt.$$

□

It remains to consider the case $1 < p \leq r$. We have the following result.

Theorem 3.5. *Let A_S^r be defined as in (2.2) and ϵ be a monotonic increasing function on $(1, \infty)$ such that $\int_1^\infty \frac{1}{t \epsilon(t)^{1/p}} dt < \infty$. If $1 < p \leq r$, then*

$$\|A_S^r(\cdot \sigma)\|_{L^p(w) \rightarrow L^p(\sigma)} \lesssim \sup_Q \langle w \rangle_Q^{\frac{1}{p}} \langle \sigma \rangle_Q^{\frac{1}{p'}} \rho_{\sigma, \epsilon}(Q)^{\frac{1}{p}}.$$

Proof. From the proof of Theorem 2.13, we know that

$$\|A_S^r(\cdot \sigma)\|_{L^p(w) \rightarrow L^p(\sigma)} \leq \sup_{R \in \mathcal{S}} \frac{\left\| \sum_{Q \in \mathcal{S}} \langle \sigma \rangle_Q \mathbf{1}_Q \right\|_{L^p(w)}}{\sigma(R)^{1/p}}.$$

Then by the same argument as that in [8], we can get the conclusion. □

Now with Theorems 3.2 and 3.5, Theorem 1.4 follows.

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